

ON THE INVERSE POWER INDEX PROBLEM

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ABSTRACT. Weighted voting games are frequently used in decision making. Each voter has a weight and a proposal is accepted if the weight sum of the supporting voters exceeds a quota. One line of research is the efficient computation of so-called power indices measuring the influence of a voter. We treat the inverse problem: Given an influence vector and a power index, determine a weighted voting game such that the distribution of influence among the voters is as close as possible to the given target value. We present exact algorithms and computational results for the Shapley-Shubik and the (normalized) Banzhaf power index.

1. INTRODUCTION

The European Economic Community, i. e. the predecessor of the European Union, was founded in 1957 by Belgium, France, Germany, Italy, Luxembourg and the Netherlands. Decisions were and are made via polling. More precisely, each country has a given voting weight, see Table 1¹, and a proposal is approved if it is supported by at least 12, called the quota, out of the 17 votes. So Luxembourg possesses $\frac{1}{17}$ of all votes. But let us consider the following thought experiment: The five countries except Luxembourg vote for their own without asking for Luxembourg's opinion. In any case the number s of supporting votes is even. Thus the vote of Luxembourg does not matter at all. For $s \geq 12$ the proposal will be approved and for $s \leq 10$ the proposal will be rejected. We may say that Luxembourg has absolutely no power in this weighted voting game.

state	votes	population (2010)
Germany	4	82144902
France	4	62582650
Italy	4	60017346
the Netherlands	2	16503473
Belgium	2	10783738
Luxembourg	1	494153

TABLE 1. Voting weights and the quota in the European Economic Community of 1957.

To have a measurement of the relative power of the members of voting schemes in more complex situations, power indices like the Shapley-Shubik or the Banzhaf power index were invented, see the next section for a definition. Table 2 lists the values of the two mentioned power indices for three different weighted voting games. We remark that there is a non-linear dependency between weights, quota and the value of the considered power index.

If we suppose that there exists some agreement on a fair distribution of power $(d_i)_{1 \leq i \leq n}$ among the n countries and a selection of an appropriate power index \mathcal{P} , we obtain the inverse power index problem²: $\min \|d - \mathcal{P}(\chi)\|$, where χ is a weighted voting game for n voters and $\|\cdot\|$ an arbitrary suitable norm to measure the deviation, i. e. a rule to decide which assignment of voting weights is *fairer*. In practice the intended power distribution d may arise from the population numbers, either in the variant “One Person, One Vote”, i. e. d_i is proportional to the population of country i , or by using Penrose square root law, i. e. d_i is proportional to the square root of the population of country i [20, 22].

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¹Recent estimates for the population are taken from <http://epp.eurostat.ec.europa.eu>.

²In [11] the authors call this problem the *voting game design problem*.

state	w_i	\mathcal{SS}	\mathcal{BZ}	w_i	\mathcal{SS}	\mathcal{BZ}	w_i	\mathcal{SS}	\mathcal{BZ}
Germany	4	0.233	0.238	4	0.233	0.229	4	0.233	0.222
France	4	0.233	0.238	4	0.233	0.229	4	0.233	0.222
Italy	4	0.233	0.238	4	0.233	0.229	4	0.233	0.222
Netherlands	2	0.150	0.143	3	0.200	0.188	2	0.100	0.111
Belgium	2	0.150	0.143	2	0.050	0.063	2	0.100	0.111
Luxembourg	1	0.000	0.000	1	0.050	0.063	1	0.100	0.111
quota	12			12			11		

TABLE 2. Power indices for different weighted voting games.

1.1. Related work. There is a vast literature how to compute the Shapley-Shubik index and other power indices in various circumstances, either exactly or approximatively. Few results are known about the inverse power index problem so far. Leech proposes a certain kind of fixed point algorithm and reports that it works considerably well in practice, whenever the number of voters n is not too small [16, 15], see also [14]. The author argues that for large n one may assume that the power index smoothly depends on the voting weights and thus Brouwer’s fixed point theorem can be applied to *prove* the convergence of this approach. Another heuristic is described in [4].

Besides using the finiteness of the set of weighted voting games, the first general bounds stating that some power distributions can not be approximated by Banzhaf vectors too closely are given by Alon and Edelman [1].

Since for each number of voters n there is only a finite set of weighted voting games one may simply solve the inverse power index problem by looping over the whole set. The enumeration of weighted voting games dates back to at least 1962 [18], where up to 6 voters are treated. For $n = 7, 8$ voters we refer e. g. to [26, 19, 12]. Bart de Keijzer presents a promising graded poset for weighted voting games in his master thesis [10], see also [11].³ $n = 9$ voters were successfully treated in [13, 23].

A similar inverse problem was solved in [3].

1.2. Our contribution. We present the first exact algorithm, besides complete enumeration of weighted voting games, for the inverse power index problem. We computationally show that for $n \leq 15$ voters the Shapley-Shubik vector of every simple game, complete simple game or weighted voting game has a $\|\cdot\|_1$ -distance to $(0.75, 0.25, 0 \dots)$ of at least $\frac{1}{3}$. For the Banzhaf power index the (conjectured) bound for the worst possible approximation in $\|\cdot\|_1$ -distance is a bit more involved. But it seems that for $n \geq 2$ voters $(0.75, 0.25, 0 \dots)$ is one desired power distribution where the worst case is attained. We present computational results for $n \leq 11$ for simple games and $n \leq 16$ for complete simple games and weighted voting games. For more realistic instances we consider subsets of the European Union and determine the exact solutions for the inverse power index problem for small numbers of voters. Additionally we present some results on the set of achievable power distributions for the Banzhaf and the Shapley Shubik power index for $n \leq 9$ voters.

1.3. Outline of the paper. In Section 2 we state the definitions of the three voting methods simple games, complete simple games and weighted voting games. We further introduce the Banzhaf and the Shapley Shubik power index. The known a priori estimates are the topic of Section 3. We give precise conjectures about the, with respect to approximation via Banzhaf and Shapley Shubik vectors, worst case power distributions. In Section 4 we give an exact algorithm for the inverse power index problem for simple games, complete simple games and weighted voting games for the Banzhaf and the Shapley Shubik power index via an ILP formulation. Another exact approach via exhaustive enumeration is described in Section 5. A promising combination of both methods as a branch&bound approach is outlined in Subsection 5.1. Computational results evaluating the proposed exact algorithms for the inverse power index problem and proving the stated conjectures for small numbers of voters are given in Section 6. To get a more global understanding we consider the sets of achievable Banzhaf and Shapley Shubik vectors in Section 7 and close with a conclusion in Section 7. Additionally we classify the complete set of desired power distributions which have the worst case approximation property for very small n in an appendix.

³We would like to remark that the counts for weighted voting games with $6 \leq n \leq 8$ voters are wrongly stated in [10], but the methods should work.

2. VOTING METHODS AND POWER INDICES

Consider a yes-no voting system for a set $N = \{1, 2, \dots, n\}$ of n voters. The acceptance of a proposal depends on the subset of the supporting voters. In general this can be described by a Boolean function $\chi : 2^N \rightarrow \{0, 1\}$. A quite natural requirement for a voting system is monotonicity, i. e. $\chi(U) \leq \chi(U')$ for all $U \subseteq U' \subseteq N$. Monotone Boolean functions $\chi : 2^N \rightarrow \{0, 1\}$ with $\chi(\emptyset) = 0$ and $\chi(N) = 1$ are called simple games.

Definition 1. For a simple game $\chi : 2^N \rightarrow \{0, 1\}$ a subset $U \subseteq N$ is called a *minimal winning coalition* if $\chi(U) = 1$ and $\chi(U \setminus \{i\}) = 0$ for all $i \in U$. Analogously a subset $U \subseteq N$ is a *maximal losing coalition* if $\chi(U) = 0$ and $\chi(U \cup \{i\}) = 1$ for all $i \in N \setminus U$.

Each simple game is completely characterized by either the complete list \overline{W} of its minimal winning coalitions or the complete list \overline{L} of its maximal losing coalitions. Both the set of minimal winning coalitions and the set of the maximal losing coalitions are antichains. For each simple game χ there is a dual simple game χ^d defined via $\chi^d(U) = 1 - \chi(N \setminus U)$. A voter $i \in N$ is called a null voter if he is not a member of at least one minimal winning coalition.

In many applications voting systems are restricted to smaller sub classes of simple games, e. g. complete simple games arising from Isbell's desirability relation [9]: We write $i \sqsupset j$ for two voters $i, j \in N$ iff we have $\chi(\{i\} \cup U \setminus \{j\}) \geq \chi(U)$ for all $j \in U \subseteq N \setminus \{i\}$. A pair (N, χ) is called complete simple game if it is a simple game and the binary relation \sqsupset is a total preorder and we abbreviate $i \sqsupset j, j \sqsupset i$ by $i \sqsubseteq j$. To factor out symmetries we assume $1 \sqsupset 2 \sqsupset \dots \sqsupset n$ and write coalitions $U \subseteq N$ as characteristic vectors u . We call u a winning vector iff $\chi(U) = 1$, otherwise we call u a losing vector.

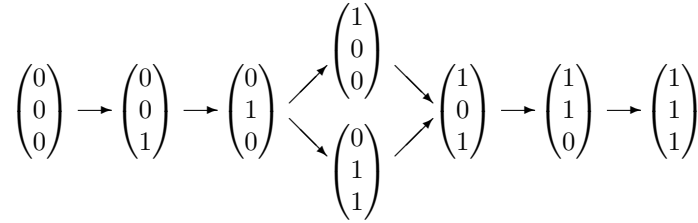


FIGURE 1. The Hasse diagram for \preceq on $N = \{1, 2, 3\}$.

To state an analogue of minimal winning coalitions for complete simple games we need another partial ordering: For two coalitions $u = (u_1 \dots u_n), v = (v_1 \dots v_n) \in \{0, 1\}^n$ we write $u \preceq v$ iff we have $\sum_{i=1}^k u_i \leq \sum_{i=1}^k v_i$ for all $1 \leq k \leq n$ and abbreviate $u \preceq v, u \neq v$ by $u \prec v, u \not\preceq v, u \not\prec v$ by $u \bowtie v$. With this we have $\chi(u) \leq \chi(v)$ for all $u \preceq v$. By \widehat{W} we denote the set of all shift-minimal winning vectors, i. e. winning vectors which are minimal with respect to \preceq . Similarly we denote by \widehat{L} the set of all shift-maximal losing vectors, i. e. losing vectors which are maximal with respect to \preceq . We remark that each complete simple game is uniquely characterized by either \widehat{W} or \widehat{L} , see e. g. [2]. In Figure 1 we depict the Hasse diagram for \preceq on $N = \{1, 2, 3\}$ and see that there are exactly 8 complete simple games (or antichains) for $n = 3$ voters; 7 with one shift-minimal winning vector and one with two shift-minimal winning vectors.

An even more important sub class of complete simple games are weighted voting games given by non-negatives weights w_i for all $i \in N$ and a quota q . The corresponding monotone Boolean function χ is obtained by

$$\chi(U) = 1 \iff \sum_{i \in U} w_i \geq q.$$

Algorithmically we can check whether a given simple game χ is a weighted voting game by solving a suitable linear program, see Section 5.

In Table 3 we state the known numbers of simple games, complete simple games and weighted voting games for n voters, see [7, 13, 23, 25]. To define two of the most important power indices for these three types of voting methods we need another definition:

Definition 2. For a voter $i \in N$ and a monotone Boolean function χ , an *i-swing* is a set $U \subseteq N \setminus \{i\}$ such that $\chi(U) = 0$ and $\chi(U \cup \{i\}) = 1$.

	n	1	2	3	4	5	6	7	8	9
#sg	1	4	18	166	7579	7828352	2414682040996	5.613043 · 10 ²²		> 10 ⁴²
#csg	1	3	8	25	117	1171		44313	16175188	284432730174
#wvg	1	3	8	25	117	1111		29373	2730164	989913344

TABLE 3. Number of simple games, complete simple games and weighted voting games for n voters.

With this we can express the Banzhaf vector $(\mathcal{BZ}(\chi, 1), \dots, \mathcal{BZ}(\chi, n))$ as

$$\mathcal{BZ}(\chi, i) = \frac{1}{m} \cdot |\{U \subseteq N \setminus \{i\} \mid U \text{ is an } i\text{-swing}\}|,$$

where m is the total number of swings. Also the Shapley-Shubik vector $(\mathcal{SS}(\chi, 1), \dots, \mathcal{SS}(\chi, n))$ can be expressed using swings via

$$\mathcal{SS}(\chi, i) = \frac{1}{n!} \cdot \sum_{U \text{ is an } i\text{-swing}} |U|! \cdot (n - |U| - 1)!.$$

Since a set U is an i -swing in a simple game χ if and only if it is an i -swing in the dual game χ^d , both the Shapley-Shubik vector and the Banzhaf vector coincide for pairs of dual simple games. For both power indices the value for a null voter is zero. Removing a null voter from a simple game with n voters gives a simple game with $n - 1$ voters, where the power indices of the remaining $n - 1$ voters coincide.

3. A PRIORI ESTIMATES

Since heuristics can only find good solutions but are not capable of providing lower bounds we are considering a priori estimates. The first fact coming to mind is the finite number of weighted voting games with n voters. An upper bound of 2^{n^2-n+2} is given in [11]. Respecting symmetry we conclude that for $n \geq 2$ there are at most $2^{n^2-n+2} \cdot n! < 2^{n^2} \cdot n^n$ different Banzhaf- or Shapley-Shubik vectors in $[0, 1]^n$. Decomposing $[0, 1]^n$ into $2^{n^2} \cdot n^n$ subcubes of side length $\frac{1}{2^{n^2} \cdot n^n}$ results in at least one empty subcube using the pigeonhole principle. Thus there are points in $[0, 1]^n$ such that the nearest Banzhaf- or Shapley-Shubik vector is at least $\frac{1}{2^{n^2} \cdot n^n}$ apart.

Having a closer look at the swings one can obtain a slightly sharper bound. For n voters the number of swings lies between n and $(\lfloor \frac{n}{2} \rfloor + 1) \cdot (\lfloor \frac{n}{2} \rfloor + 1)$, see [5]. Using the fact that the numbers η_i of swings for voter i all have the same parity we can conclude that either $\mathcal{BZ}(\chi, i) = \mathcal{BZ}(\chi, j)$ or we have

$$|\mathcal{BZ}(\chi, i) - \mathcal{BZ}(\chi, j)| \geq \frac{2}{(\lfloor \frac{n}{2} \rfloor + 1) \cdot (\lfloor \frac{n}{2} \rfloor + 1)}$$

for voters i and j .

Both bounds tend to zero when n tends to infinity. Recently Alon and Edelman [1] found a lower bound that does not tend to zero:

Theorem 1. *Let $n > k$ be positive integers, let $\varepsilon < \frac{1}{k+1}$ be a positive real, and let χ be a simple game for n voters.*

If $\sum_{i=k+1}^n \mathcal{BZ}(\chi, i) \leq \varepsilon$, then there exists a simple game χ' for k voters such that

$$\|B(\chi) - B(\chi')\|_1 = \sum_{i=1}^k |\mathcal{BZ}(\chi, i) - \mathcal{BZ}(\chi', i)| + \sum_{i=k+1}^n |\mathcal{BZ}(\chi, i) - 0| \leq \frac{(2k+1)\varepsilon}{1 - (k+1)\varepsilon} + \varepsilon.$$

Alon and Edelman have chosen $k = 2$ and $\varepsilon = 0.01$ to deduce that for a simple game with $\sum_{i=3}^n \mathcal{BZ}(\chi, i) \leq 0.01$ the two-dimensional vector $(\mathcal{BZ}(\chi, 1), \mathcal{BZ}(\chi, 2))$ lies within $\|\cdot\|_1$ -distance smaller than $\frac{1}{16}$ of one of the achievable Banzhaf vectors, i. e. $(1, 0)$, $(0, 1)$, or $(\frac{1}{2}, \frac{1}{2})$. Thus for $n \geq 2$ voters the Banzhaf vector of every simple game has an $\|\cdot\|_1$ -distance of at least $\min(0.5 - \frac{1}{16}, 0.02) = 0.02$ to the power distribution $(0.75, 0.25, 0, \dots)$.

We may improve this argument a bit by choosing $k = 2$ and $\varepsilon = \frac{1}{18}$ to deduce that the minimal $\|\cdot\|_1$ -distance to the power distribution $(0.75, 0.25, 0, \dots)$ is at least $\frac{1}{9}$, which is best possible using the general theorem.

Conjecture 1. For $n \geq 2$ voters consider the following desired power distribution: $d_1 = 0.75$, $d_2 = 0.25$, and $d_i = 0$ for $3 \leq i \leq n$. For each simple game (complete simple game or weighted voting game) χ we have the inequalities $\|\mathcal{SS}(\chi) - d\|_1 = \sum_{i=1}^n |\mathcal{SS}(\chi, i) - d_i| \geq \frac{1}{3}$ and $\|\mathcal{BZ}(\chi) - d\|_1 = \sum_{i=1}^n |\mathcal{BZ}(\chi, i) - d_i| \geq \frac{k_m}{l_m}$, where $k_1 = 1$, $k_m = \begin{cases} 2k_{m-1} & : m \equiv 0 \pmod{2}, \\ 8k_{m-1} - 1 & : m \equiv 1 \pmod{2} \end{cases}$ for $m \geq 2$ and $l_1 = 2$, $l_2 = 5$, $l_m = \begin{cases} 2l_{m-1} + 3 & : m \equiv 0 \pmod{2}, \\ 8l_{m-1} - 2 & : m \equiv 1 \pmod{2} \end{cases}$ for $m \geq 3$.

m	2	3	4	5	6	7
$\frac{k_m}{l_m}$	$\frac{1}{2} = 0.5$	$\frac{2}{5} = 0.4$	$\frac{15}{18} \approx 0.39474$	$\frac{30}{79} \approx 0.37975$	$\frac{239}{630} \approx 0.37937$	$\frac{478}{1263} \approx 0.37846$

TABLE 4. Values of $\frac{k_m}{l_m}$ – the conjectured lower bound for the Banzhaf index.

Using $\tau(n) = \frac{(-1)^{n+1}+1}{2}$ a closed expression for these numbers $\frac{k_m}{l_m}$, see also Table 4, is given by $k_m = \frac{7 \cdot 2^{2m-2+\tau(m)} + 2 - \tau(m)}{15}$ and $l_m = 2^{2m-3+\tau(m)} + \frac{11 \cdot 2^{2m-2+\tau(m)} + 1 - 23\tau(m)}{15}$ for $m \geq 2$. Thus we have $\lim_{m \rightarrow \infty} \frac{k_m}{l_m} = \frac{14}{37} \approx 0.378378 > \frac{1}{3}$.

Conjecture 1 looks rather specific. A first hint why $d_1 = 0.75$, $d_2 = 0.25$ might be hard to approximate is the fact that this point maximizes the $\|\cdot\|_1$ -distance to the achievable power vectors $(1, 0)$, $(0.5, 0.5)$, $(0, 1)$, both for the Shapley-Shubik and the Banzhaf index. For some small $n > 2$ we compute the points or regions which maximize the $\|\cdot\|_1$ -distance to the achievable power vectors in Appendix A. So far we have not seen a geometric structure behind the values $\frac{k_m}{l_m}$ for the Banzhaf index, but computing these numbers for simple games and weighted voting games with $n \leq 16$ suggests that the rather structured recursion formulas for k_m and l_m might be more than a pure coincidence. Having Theorem 1 in mind one might conjecture that power distributions with many zeros are hard to approximate whenever they are hard to approximate without the zero values. A first step towards understanding the inverse power index problem might be the results on achievable hierarchies in simple games, see [8].

Additionally we conjecture that there is a similar result as Theorem 1 for the Shapley-Shubik index. In the next sections we will present some exact algorithms to solve the inverse power index problem and use them to prove Conjecture 1 for small n .

Conjecture 2. For each desired power distribution d_i with $d_i \geq 0$ and $\sum_{i=1}^n d_i = 1$, where $n \geq 3$, there exist two weighted voting games (complete simple games or simple games) χ, χ' with $\|\mathcal{BZ}(\chi) - d\|_1 = \sum_{i=1}^n |\mathcal{BZ}(\chi, i) - d_i| \leq \frac{k_m}{l_m}$ and $\|\mathcal{SS}(\chi') - d\|_1 = \sum_{i=1}^n |\mathcal{SS}(\chi', i) - d_i| \leq \frac{1}{3}$.

4. AN ILP FORMULATION FOR THE INVERSE POWER INDEX PROBLEM

Our first exact approach to solve the inverse power index problem is to utilize an ILP formulation using `ILLOG CPLEX 11` as an ILP solver. Therefore let n be the number of voters and $d = (d_1, \dots, d_n)$ the desired power distribution with $d_i \geq 0$ and $\sum_{i=1}^n d_i = 1$. To model the monotone Boolean function χ we use binary variables x_U which equal $\chi(U)$ for all subsets $U \subseteq N$. In order to express the values $\mathcal{SS}(\chi, i)$ and $\mathcal{BZ}(\chi, i)$ of the power indices for voter i we additionally introduce binary variables $y_{i,U}$ which equal one iff U is an i -swing, see Equation (5). With this we can state the Shapley-Shubik value for voter i as

$$\mathcal{SS}(i) = p_i = \frac{1}{n!} \cdot \sum_{U \subseteq N \setminus \{i\}} (|U|! \cdot (n - |U| - 1)!) \cdot y_{i,U}$$

and the number of i -swings as $s_i = \sum_{U \subseteq N \setminus \{i\}} y_{i,U}$ for all $1 \leq i \leq n$. To measure the deviation $|p_i - d_i|$ for voter i we introduce variables δ_i and obtain the following binary ILP formulation for the inverse Shapley-Shubik index

problem with $\|\cdot\|_1$ -norm on simple games:

$$\begin{aligned}
(1) \quad & \min && \sum_{i=1}^n \delta_i \\
(2) \quad & s.t. && p_i \leq d_i + \delta_i && \forall 1 \leq i \leq n, \\
(3) \quad & && p_i \geq d_i - \delta_i && \forall 1 \leq i \leq n, \\
(4) \quad & && p_i = \frac{1}{n!} \cdot \sum_{U \subseteq N \setminus \{i\}} (|U|! \cdot (n - |U| - 1)!) \cdot y_{i,U} && \forall 1 \leq i \leq n, \\
(5) \quad & && y_{i,U} = x_{U \cup \{i\}} - x_U && \forall 1 \leq i \leq n, U \subseteq N \setminus \{i\}, \\
(6) \quad & && x_U \geq x_{U \setminus \{j\}} && \forall \emptyset \neq U \subseteq N, j \in U, \\
(7) \quad & && x_\emptyset = 0 \\
(8) \quad & && x_N = 1 \\
(9) \quad & && x_U \in \{0, 1\} && \forall U \subseteq N, \\
(10) \quad & && y_{i,U} \in \{0, 1\} && \forall 1 \leq i \leq n, U \subseteq N \setminus \{i\}, \\
(11) \quad & && \delta_i \geq 0 && \forall 1 \leq i \leq n, \\
(12) \quad & && p_i \geq 0 && \forall 1 \leq i \leq n.
\end{aligned}$$

Of course we may directly eliminate all variables $y_{i,U}$ and p_i from this formulation. The equations and inequalities (6), (7), (8), and (9) enforce that χ is a simple game. To restrict χ to complete simple games only small adjustments are necessary: We replace the subsets $U \subseteq N$ by its characteristic vectors $u \in \{0, 1\}^n$ and rewrite Inequality (6) to $x_u \leq x_v$ for all $u, v \in \{0, 1\}^n$ with $u \preceq v$, or more economically to

$$(13) \quad x_u \leq x_v \quad \forall v \in \{0, 1\}^n, u \in \rightarrow v : \|v\|_1 \neq 0,$$

where $\rightarrow v$ denotes the set of elements arising from v by a right-shift of one of the rightmost 1s in v . Examples are given by $\rightarrow (1, 0, 1, 0) = \{(1, 0, 0, 1), (0, 1, 1, 0)\}$ and $\rightarrow (1, 0, 1) = \{(1, 0, 0), (0, 1, 1)\}$.

To restrict χ to weighted voting games we additionally have to introduce weights $w_i \geq 0$ and a quota $q > 0$. To be able to decide whether $\sum_{i \in U} w_i \geq q$ or $\sum_{i \in U} w_i < q$ in the framework of ILPs we restrict ourselves to weights w_i which differ by at least one whenever they are different. (We may simply use integer weights, which could result in harder problems for the ILP solver.) To interlink the x_U with the w_i and q we use

$$(14) \quad q - (1 - x_U) \cdot M - \sum_{i \in U} w_i \leq 0 \quad \forall U \subseteq N$$

and

$$(15) \quad -x_U \cdot M + \sum_{i \in U} w_i \leq q - 1 \quad \forall U \subseteq N,$$

where M is a suitably large constant fulfilling $M - 1 \geq \sum_{i=1}^n w_i$. (We may choose $M = 4n \left(\frac{n+1}{4}\right)^{(n+1)/2}$, see [17, Theorem 9.3.2.1].) Now let us argue why Inequalities (14) and (15) are necessary and sufficient. If $x_U = 1$ in (14) and $x_U = 0$ in (15) then the inequalities are equivalent to $\sum_{i \in U} w_i \geq q$ and $\sum_{i \in U} w_i \leq q - 1$, respectively. Otherwise

(for $x_U \in \{0, 1\}$) both sets of inequalities are valid due to $M - 1 \geq \sum_{i=1}^n w_i$. So we have given ILP formulations for the considered three types of voting methods when the Shapley-Shubik index and the $\|\cdot\|_1$ -norm is used.

For the Banzhaf index we have to tackle the fact that its value for voter i is given by the following non-linear equality $\mathcal{BZ}(i) = \frac{s_i}{\sum_{j=1}^n s_j}$ and we have to evaluate the deviation $\sum_{i=1}^n |\mathcal{BZ}(i) - d_i| = \left| \frac{s_i}{\sum_{j=1}^n s_j} - d_i \right|$.

To this end we set $s = \sum_{i=1}^n s_i$ and $\delta_i = |s_i - d_i \cdot s|$. Instead of minimizing the sum of the δ_i we introduce $\sum_{i=1}^n \delta_i \leq \alpha \cdot s$, where $\alpha \in (0, 2]$ is a properly chosen constant. With this we can state: If the following binary ILP has a solution then the corresponding simple game χ approximates the desired power distribution d with an error in the $\|\cdot\|_1$ -norm of at most α . Otherwise no such approximation is possible. Thus we can minimize the deviation in the $\|\cdot\|_1$ -norm by performing a bisection on α . (Since s lies between n and $(\lfloor \frac{n}{2} \rfloor + 1) \cdot \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$, each two unequal Banzhaf vectors differ by at least $(\frac{1}{n2^n})^2$, and we only need $O(n)$ such steps.)

$$(16) \quad \sum_{i=1}^n s_i = s,$$

$$(17) \quad \sum_{i=1}^n \delta_i \leq \alpha \cdot s,$$

$$(18) \quad s_i \leq d_i \cdot s + \delta_i \quad \forall 1 \leq i \leq n,$$

$$(19) \quad s_i \geq d_i \cdot s - \delta_i \quad \forall 1 \leq i \leq n,$$

$$(20) \quad s_i = \sum_{U \subseteq N \setminus \{i\}} y_{i,U} \quad \forall 1 \leq i \leq n,$$

$$(21) \quad y_{i,U} = x_{U \cup \{i\}} - x_U \quad \forall 1 \leq i \leq n, U \subseteq N \setminus \{i\},$$

$$(22) \quad x_U \geq x_{U \setminus \{j\}} \quad \forall \emptyset \neq U \subseteq N, j \in U,$$

$$(23) \quad x_\emptyset = 0$$

$$(24) \quad x_N = 1$$

$$(25) \quad x_U \in \{0, 1\} \quad \forall U \subseteq N,$$

$$(26) \quad y_{i,U} \in \{0, 1\} \quad \forall 1 \leq i \leq n, U \subseteq N \setminus \{i\},$$

$$(27) \quad \delta_i \geq 0 \quad \forall 1 \leq i \leq n,$$

$$(28) \quad s_i \geq 0 \quad \forall 1 \leq i \leq n.$$

Of course we can apply similar adjustments for this ILP formulation as we did for the one modeling the Shapley-Shubik index to restrict the voting method χ , described by the x_U , to complete simple games or weighted voting games.

4.1. Simplification for Conjecture 1. In order to computationally prove Conjecture 1 for the Shapley-Shubik index and small n we remove some variables and constraints from the previous ILP formulation. Since for the power distribution $d = (0.75, 0.25, 0 \dots)$ we have $\sum_{i=1}^n \delta_i = d_1 + d_2 + (1 - p_1 - p_2)$ we can delete all variables and constraints for $i \geq 3$:

$$\begin{aligned} \min \quad & d_1 + d_2 + 1 - p_1 - p_2 \\ \text{s.t.} \quad & p_i \leq d_i + \delta_i \quad \forall 1 \leq i \leq 2, \\ & p_i \geq d_i - \delta_i \quad \forall 1 \leq i \leq 2, \\ & p_i = \frac{1}{n!} \cdot \sum_{U \subseteq N \setminus \{i\}} (|U|! \cdot (n - |U| - 1)!) \cdot y_{i,U} \quad \forall 1 \leq i \leq 2, \\ & y_{i,U} = x_{U \cup \{i\}} - x_U \quad \forall 1 \leq i \leq 2, U \subseteq N \setminus \{i\}, \\ & x_U \geq x_{U \setminus \{j\}} \quad \forall \emptyset \neq U \subseteq N, U \cap \{1, 2\} \neq \emptyset, j \in U, \\ & x_\emptyset = 0, \\ & x_N = 1, \\ & x_U \in \{0, 1\} \quad \forall U \subseteq N, \\ & y_{i,U} \in \{0, 1\} \quad \forall 1 \leq i \leq 2, U \subseteq N \setminus \{i\}, \\ & \delta_1, \delta_2 \geq 0, \\ & p_1, p_2 \geq 0. \end{aligned}$$

5. AN EXHAUSTIVE ENUMERATION APPROACH FOR THE INVERSE POWER INDEX PROBLEM

The conceptionally easiest exact algorithm to solve the inverse power index problem is to loop over the finite set of all voting methods and to pick that one that minimizes the deviation between the desired and the achieved power distribution. Table 3 proves this approach to be infeasible for simple games and $n \geq 8$. For complete simple games there exists a very effective parameterization, see [2]. But, to our knowledge, nobody has determined the number of complete simple games for $n \geq 10$ yet. On the one hand the fraction of complete simple games

which are weighted voting games tends to zero as the number of voters tends to infinity, but on the other hand no parameterization of weighted voting games is known. Several recursive constructions, mostly using super classes of weighted voting games, are presented in the literature [11, 12, 7, 19]. Here we will describe the approach of [7, 6] which exhaustively generates the whole class of complete simple games in the first run and afterwards checks each complete simple game whether it is weighted or not via a linear program, see also [24, 13]. One variant is given by

$$(29) \quad \sum_{i=1}^n u_i w_i \geq q \quad \forall u \in \widehat{W},$$

$$(30) \quad 1 + \sum_{i=1}^n v_i w_i \leq q \quad \forall v \in \widehat{L},$$

$$(31) \quad w_i \geq w_{i+1} \quad \forall 1 \leq i \leq n-1,$$

$$(32) \quad w_n, q \geq 0.$$

For $n = 9$ voters it seems inconceivable to solve 284432730174 linear programs in order to determine all weighted voting games, nevertheless this is – more or less – what was done in [23]⁴. A more effective way is described in [13] and uses partial complete simple games.

As mentioned before a complete simple game can be characterized by its set \widehat{W} of shift-minimal winning vectors. Now we want to exhaustively generate complete simple games for a given number of voters using an orderly generation approach, see [21], i. e. we start with an empty set W and add shift-minimal winning vectors, which are decreasing with respect to the lexicographical ordering \leq_{lex} ⁵, step by step. In the corresponding generation tree for a node W all successors W' fulfill $W \subseteq W'$ and $y <_{\text{lex}} x$ for all $x \in W, y \in W' \setminus W$. Now let L and L' be the set of all shift-maximal losing coalitions corresponding to W and W' . For a given set $W \neq \emptyset$ with lexicographically smallest element w we define $\tilde{L} := \left\{ v \in \mathbb{B}^n \mid \exists u \in W : v \prec u \right\} \cup \left\{ v \in \mathbb{B}^n \mid \nexists u \in W \cup \{x \in \mathbb{B}^n \mid x <_{\text{lex}} w\} : v \succeq u \right\}$, where $\mathbb{B} = \{0, 1\}$ and condense the shift-maximal vectors of \tilde{L} to a set $\dot{L} \subseteq \tilde{L}$. With this we have $\dot{L} \subseteq L$ and $\dot{L} \subseteq L'$ for all successors of W . Thus whenever the previous linear program where \widehat{W} is replaced by W and \widehat{L} is replaced by \dot{L} is infeasible, we can prune the whole search tree below node W .

Let us have an example for $n = 7$ voters to illustrate the proposed algorithm. For the partial set $W = \{1110000, 1101010, 1001111\}$ of shift-minimal winning vectors we can determine the partial set

$$\dot{L} = \{1101000, 1011010, 1100110, 1101001, 0101111\}$$

of shift-maximal losing vectors. Here 1101000 is an element of \dot{L} since we have $1101000 \prec 1110000 \in W$. Since the linear program corresponding to W and \dot{L} is infeasible we know that none of the complete simple games given by the set $W \cup X$ of shift-maximal winning vectors is weighted, where

$$X \subset \{0101111, 0101110, 0101101, 0011111, 0011110, 0011101\}.$$

So at least 7 complete games are shown to be non-weighted by solving one linear program.

Using the concept of orderly generation and partial complete games drastically reduces computation times. However the number of weighted voting games for $n = 10$ voters remains unknown. So this approach is limited to rather small numbers of voters.

5.1. A branch&bound approach for the inverse power index problem for weighted voting games. Since we will see in Section 6 that the LP relaxation of our ILP models from Section 4 is rather weak we propose a branch&bound algorithm based on the discrete structure of partial complete games.

As in Section 5 we start with an empty set W of the shift-minimal winning vectors. Again we generate complete simple games via orderly generation and prune the subtree of W whenever W can not be extended to a weighted voting game. To further prune the search tree we additionally use the information of W and the partial set \dot{L} of shift-maximal losing vectors to determine some of the x_U -values in the ILPs of Section 4. We need a computationally cheap lower bound on the deviation between the values of the given power index for all voting methods χ below node W and the desired power distribution d . If the best known solution so far is better than this lower bound, we can prune the search tree. A quite natural candidate for such a lower bound is the LP relaxation of our ILP

⁴Additionally some computational tricks and heuristics were used to reduce the number of linear programs.

⁵Here one may also read the coalition vectors as integers written in their binary expansion and use the ordinary ordering \leq of integers.

models from Section 4. To know good solutions right in the beginning of our branch&bound tree we may utilize any heuristics for the inverse power index problem. In this paper we consider Leech's algorithm.

6. COMPUTATIONAL RESULTS

state	pop.	state	pop.	state	pop.
Austria	8200	Germany	82500	Netherlands	16300
Belgium	10400	Greece	11100	Poland	38200
Bulgaria	7800	Hungary	10200	Portugal	10100
Cyprus	700	Ireland	4100	Romania	21700
Czech Republic	10500	Italy	58500	Slovakia	5400
Denmark	5400	Latvia	3400	Slovenia	2000
Estonia	1300	Lithuania	2300	Spain	43000
Finland	5200	Luxembourg	500	Sweden	9000
France	60000	Malta	400	United Kingdom	60600

TABLE 5. Population (in thousands) of the 27 member states of the European Union.

inst.	simple games			complete simple games		
	$\ \cdot\ _1$	nodes	time	$\ \cdot\ _1$	nodes	time
EU ₁	$0.00000 \cdot 10^{-0}$	1	0.01s	$0.00000 \cdot 10^{-0}$	1	0.01s
EU ₂	$7.69740 \cdot 10^{-2}$	1	0.01s	$7.69740 \cdot 10^{-2}$	1	0.01s
EU ₃	$7.13793 \cdot 10^{-2}$	5	0.05s	$7.13793 \cdot 10^{-2}$	4	0.01s
EU ₄	$6.30740 \cdot 10^{-2}$	28	0.01s	$6.30740 \cdot 10^{-2}$	9	0.02s
EU ₅	$6.90250 \cdot 10^{-2}$	303	0.09s	$6.90250 \cdot 10^{-2}$	26	0.04s
EU ₆	$4.18923 \cdot 10^{-2}$	18265	5.91s	$5.40190 \cdot 10^{-2}$	35	0.11s
EU ₇	$2.39402 \cdot 10^{-2}$	71483741	15.88h	$3.75078 \cdot 10^{-2}$	54	0.41s
EU ₈				$1.78178 \cdot 10^{-2}$	210	3.08s
EU ₉				$6.89922 \cdot 10^{-3}$	1271	95.91s
EU ₁₀				$3.65220 \cdot 10^{-3}$	19520	101.21m
EU ₁₁				$1.31318 \cdot 10^{-3}$	349125	55.25h

TABLE 6. Results for the EU using the Shapley-Shubik power index.

As a realistic input instance for the inverse power index problem we use the 27 countries of the European Union with population numbers pop_i given in Table 5. Instance EU_n consists of the n countries with the largest population, where the desired power is given by

$$d_i = \frac{\sqrt{pop_i}}{\sum_{j=1}^n \sqrt{pop_j}},$$

i. e. we apply Penrose square root law. In Table 6 and Table 7 we state the results for simple games, complete simple games, and weighted voting games, respectively, using the Shapley-Shubik power index.

The weighted voting games corresponding to Table 7 can be represented by their quota and their weights yielding: $[1; 1]$, $[2; 1, 1]$, $[2; 1, 1, 1]$, $[3; 1, 1, 1, 1]$, $[4; 1, 1, 1, 1, 1]$, $[14; 5, 5, 4, 4, 3, 3]$, $[18; 9, 8, 7, 7, 6, 5, 4]$, $[41; 13, 11, 11, 10, 9, 9, 7, 6]$, $[92; 30, 27, 27, 26, 23, 22, 17, 16, 14]$, $[109; 34, 29, 29, 28, 25, 23, 18, 16, 14, 13]$, and $[339; 96, 82, 81, 80, 71, 65, 51, 44, 38, 36, 36]$.

For $n \leq 5$ the optimal solutions for simple games and complete simple games coincide. For larger n simple games admit better approximations but require a huge amount of cpu time. For $n \leq 9$ the optimal solutions for complete simple games and weighted voting games coincide. We would like to remark that for weighted voting games the Inequalities (14) and (15) would suffice if the y -variables are binaries but adding the valid Inequalities (13) drastically reduces the necessary cpu time.

We would like to remark that our ILP formulations for the inverse power index problem have a large integrality gap so that the lower bound stays zero for a long time during the solution process. Being more precisely we state

inst.	weighted voting games			
	$\ \cdot\ _1$	nodes	time	lower bound
EU ₁	$0.00000 \cdot 10^{-0}$	1	0.01s	$0.00000 \cdot 10^{-0}$
EU ₂	$7.69740 \cdot 10^{-2}$	3	0.01s	$7.69740 \cdot 10^{-2}$
EU ₃	$7.13793 \cdot 10^{-2}$	9	0.01s	$7.13793 \cdot 10^{-2}$
EU ₄	$6.30740 \cdot 10^{-2}$	5	0.02s	$6.30740 \cdot 10^{-2}$
EU ₅	$6.90250 \cdot 10^{-2}$	23	0.04s	$6.86700 \cdot 10^{-3}$
EU ₆	$5.40190 \cdot 10^{-2}$	29	0.17s	$2.52257 \cdot 10^{-2}$
EU ₇	$3.75078 \cdot 10^{-2}$	44	0.40s	$4.97897 \cdot 10^{-3}$
EU ₈	$1.78178 \cdot 10^{-2}$	165	2.86s	$2.20380 \cdot 10^{-3}$
EU ₉	$6.89924 \cdot 10^{-3}$	1449	49.65s	$7.19156 \cdot 10^{-4}$
EU ₁₀	$5.35622 \cdot 10^{-3}$	27517	28.68m	$1.12706 \cdot 10^{-3}$
EU ₁₁	$3.62854 \cdot 10^{-3}$	676774	69.16h	$1.14349 \cdot 10^{-4}$

TABLE 7. Results for the EU using the Shapley-Shubik power index.

n	1	2	3	4	5	6	7	8
LP-bound	0	0	0	0	$1 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$	0
n	9	10	11	12	13	14	15	
LP-bound	$9 \cdot 10^{-7}$	$8 \cdot 10^{-7}$	$1 \cdot 10^{-7}$	$3 \cdot 10^{-7}$	$2 \cdot 10^{-7}$	$8 \cdot 10^{-7}$	$6 \cdot 10^{-7}$	

TABLE 8. Lower LP-bounds for the instances EU_n for simple games, complete simple games, and weighted voting games.

the lower LP-bounds for the instances EU_n for $1 \leq n \leq 15$ players. It turns out that these bounds coincide for simple games, complete simple, and weighted voting games, but there are differences in the necessary computation times. The simple games for $n \geq 13$ took 14 seconds, 1 minute, and 6 minutes, respectively. For complete simple games the computation times for $n \geq 13$ are 2 seconds, 8 seconds, and 4 minutes. Solving the linear programs for weighted voting games with $n \geq 13$ took substantially longer: 4 minutes, 3 minutes and 55 minutes. For the instances $hard_n$, see below, we remark that for all considered voting methods and $2 \leq n \leq 16$ voters the lower LP-bound was given by zero and that we have observed a similar behavior of the computation times.

Since the given LP formulation is that weak, in terms of the integrality gap, we might use the general restrictions on the achievable power vectors from Section 3. Another possibility to obtain a lower bound is to drop the x -variables from our ILP formulations and to aggregate the $y_{i,U}$ -variables to $z_{i,|U|}$ -variables:

$$\begin{aligned}
(33) \quad & \min \quad \sum_{i=1}^n \delta_i \\
& s.t. \quad d_i - \delta_i \leq p_i \leq d_i + \delta_i \quad \forall 1 \leq i \leq n, \\
& \quad n! \cdot p_i = \sum_{j=0}^{\lceil \frac{n-2}{2} \rceil} j! \cdot (n-j-1)! \cdot y_{i,j} \quad \forall 1 \leq i \leq n, \\
& \quad \sum_{i=1}^n p_i = 1, \\
& \quad z_{i,j} \leq 2 \cdot \binom{n-1}{j} \quad \forall 1 \leq i \leq n, \forall 0 \leq j \leq \left\lfloor \frac{n-2}{2} \right\rfloor, \\
& \quad z_{i, \frac{n-1}{2}} \leq \binom{n-1}{\frac{n-1}{2}} \quad \forall 1 \leq i \leq n, n \equiv 1 \pmod{2}, \\
& \quad z_{i,j} \in \mathbb{N} \quad \forall 1 \leq i \leq n, \forall 0 \leq j \leq \left\lfloor \frac{n-2}{2} \right\rfloor, \\
& \quad \delta_i \geq 0 \quad \forall 1 \leq i \leq n.
\end{aligned}$$

Although the resulting lower bound, see the sixth column in Table 7 ⁶, is valid even for simple games, it turns out to be very useful, at least for small n . If we drop Equation (33) the ILP decomposes into n simpler ILPs which can be solved more directly, e. g. via enumeration.

inst.	simple games			complete simple games		
	$\ \cdot\ _1$	nodes	time	$\ \cdot\ _1$	nodes	time
hard ₂	0.5000	1	0.01s	0.5000	1	0.01s
hard ₃	0.3333	1	0.07s	0.3333	1	0.07s
hard ₄	0.3333	6	0.05s	0.3333	7	0.01s
hard ₅	0.3333	14	0.02s	0.3333	10	0.02s
hard ₆	0.3333	28	0.09s	0.3333	22	0.03s
hard ₇	0.3333	31	0.11s	0.3333	35	0.05s
hard ₈	0.3333	33	0.19s	0.3333	43	0.12s
hard ₉	0.3333	43	0.73s	0.3333	54	0.33s
hard ₁₀	0.3333	69	2.30s	0.3333	127	0.78s
hard ₁₁	0.3333	183	21.31s	0.3333	126	2.85s
hard ₁₂	0.3333	177	170.56s	0.3333	207	18.62s
hard ₁₃	0.3333	151	761.24s	0.3333	221	141.79s
hard ₁₄	0.3333	2354	4.32h	0.3333	194	12.52m
hard ₁₅	0.3333	3309	23.49h	0.3333	353	1.22h
hard ₁₆	0.3333			0.3333	829	53.67h

TABLE 9. Results for the desired power distribution $(0.75, 0.25, 0, \dots)$ using the Shapley-Shubik power index.

Following Conjecture 1 and 2 for $n \geq 2$ we consider the instances $hard_n$ corresponding to the desired power distribution $d = (0.75, 0.25, 0, \dots)$. In Table 9 and Table 10 we list the results for simple games, complete simple games, and weighted voting games, respectively, using the Shapley-Shubik power index.

inst.	weighted voting games		
	$\ \cdot\ _1$	nodes	time
hard ₂	0.5000	1	0.01s
hard ₃	0.3333	3	0.02s
hard ₄	0.3333	6	0.02s
hard ₅	0.3333	13	0.03s
hard ₆	0.3333	19	0.05s
hard ₇	0.3333	31	0.08s
hard ₈	0.3333	36	0.17s
hard ₉	0.3333	60	0.43s
hard ₁₀	0.3333	101	1.47s
hard ₁₁	0.3333	153	4.05s
hard ₁₂	0.3333	208	41.09s
hard ₁₃	0.3333	143	252.71s
hard ₁₄	0.3333	339	51.14m
hard ₁₅	0.3333	579	6.73h
hard ₁₆	0.3333	1087	36.80h

TABLE 10. Results for the desired power distribution $(0.75, 0.25, 0, \dots)$ using the Shapley-Shubik power index.

In Table 11 and Table 12 we list the results for simple games, complete simple games, and weighted voting games, respectively, using the Banzhaf power index.

⁶We may also add inequalities to ensure that all numbers of swings for a given voter have the same parity.

inst.	simple games			complete simple games		
	$\ \cdot\ _1$	nodes	time	$\ \cdot\ _1$	nodes	time
hard ₂	0.5000000	1	0.01s	0.5000000	1	0.01s
hard ₃	0.4000000	6	0.01s	0.4000000	5	0.07s
hard ₄	0.4000000	11	0.01s	0.4000000	9	0.17s
hard ₅	0.3947368	36	0.02s	0.3947368	13	0.25s
hard ₆	0.3947368	46	0.10s	0.3947368	21	0.26s
hard ₇	0.3797468	177	0.27s	0.3797468	15	0.18s
hard ₈	0.3797468	1923	7.80s	0.3797468	55	1.15s
hard ₉	0.3793651	11801	82.46s	0.3793651	61	2.42s
hard ₁₀	0.3793651	167868	57.60s	0.3793651	61	17.11s
hard ₁₁	0.3784640	311199	140.85m	0.3784640	110	131.67s
hard ₁₂				0.3784640	119	18.07m
hard ₁₃				0.3784399	119	87.85m
hard ₁₄				0.3784399	191	9.48h
hard ₁₅				0.3783563	217	23.78h
hard ₁₆				0.3783563	248	6.84d

TABLE 11. Results for the desired power distribution $(0.75, 0.25, 0, \dots)$ using the Banzhaf power index.

inst.	weighted voting games			
	$\ \cdot\ _1$	nodes	time	representation
hard ₂	0.5000000	1	0.01s	[1;1,0]
hard ₃	0.4000000	6	0.17s	[3;2,1,1]
hard ₄	0.4000000	26	0.16s	[3;2,1,1,0]
hard ₅	0.3947368	18	0.17s	[5;4,1,1,1,1]
hard ₆	0.3947368	45	0.28s	[5;4,1,1,1,1,0]
hard ₇	0.3797468	38	0.62s	[14;11,3,3,3,2,1,1]
hard ₈	0.3797468	55	1.41s	[14;11,3,3,3,2,1,1,0]
hard ₉	0.3793651	50	4.00s	[33;26,7,7,7,5,2,2,1,1]
hard ₁₀	0.3793651	73	24.20s	[33;26,7,7,7,5,2,2,1,1,0]
hard ₁₁	0.3784640	138	140.95s	[80;63,17,17,17,12,5,5,2,2,1,1]
hard ₁₂	0.3784640	108	26.64m	[80;63,17,17,17,12,5,5,2,2,1,1,0]
hard ₁₃	0.3784399	111	75.33m	[193;152,41,41,41,29,12,12,5,5,2,2,1,1]
hard ₁₄	0.3784399	153	12.30h	[193;152,41,41,41,29,12,12,5,5,2,2,1,1,0]
hard ₁₅	0.3783563	151	1.31d	[466;367,99,99,99,70,29,29,12,12,5,5,2,2,1,1]
hard ₁₆	0.3783563	116	2.34d	[466;367,99,99,99,70,29,29,12,12,5,5,2,2,1,1,0]

TABLE 12. Results for the desired power distribution $(0.75, 0.25, 0, \dots)$ using the Banzhaf power index.

7. POWER DISTRIBUTIONS WHICH ARE HARD TO APPROXIMATE

In the previous section we have computationally shown that the desired power distribution $(0.75, 0.25, 0, \dots)$ is hard to approximate by Shapley-Shubik or Banzhaf power vectors. Since it is reported that the proposed heuristics for the inverse power index problem perform well on practical instances one may conjecture that achievable power vectors are dense in those regions which are of practical importance and that the bad approximation property of $(0.75, 0.25, 0, \dots)$ is a very singular phenomenon.

To shed some light on this matter we will consider two dimensional projections of the set of all achievable power distributions for small n . For $1 \leq i < j \leq n$ and a (normalized) power index \mathcal{P} we define $\mathcal{A}_{i,j}^{\mathcal{P}} =$

$$\{(x, y) \mid \exists \text{ wvg } \chi \text{ consisting of } n \text{ ordered voters with } \mathcal{P}(i) = x, \mathcal{P}(j) = y\}.$$

to be the set of all achievable power vectors for ordered weighted voting games consisting of n voters and a given power index \mathcal{P} . By ordered we mean that voter i is as least as powerful as voter $i + 1$ for all $1 \leq i < n$.

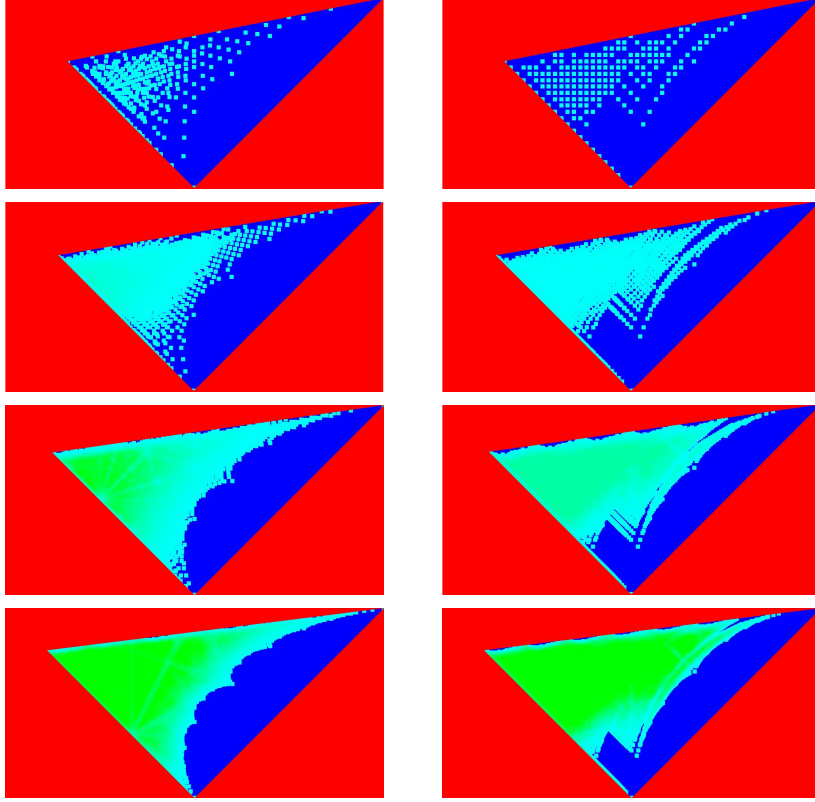


FIGURE 2. Projections to the first two voters of achievable power distributions for $6 \leq n \leq 9$ voters and the power indices Banzhaf and Shapley Shubik, i. e. the sets $\mathcal{A}_{1,2}^{\mathcal{P}}$ for $\mathcal{P} \in \{\mathcal{BZ}, \mathcal{SS}\}$.

Lemma 1. For $(x, y) \in \mathcal{A}_{i,j}^{\mathcal{P}}$ with $1 \leq i \leq n$ we have

$$\begin{aligned} 1 &\geq x \geq y \geq 0, \\ (j-1) \cdot x + (n-j+1) \cdot y &\geq 1 \text{ for } i = 1, \text{ and} \\ i \cdot x + (j-i) \cdot y &\leq 1. \end{aligned}$$

Proof. Since we assume $\mathcal{P}(1) \geq \mathcal{P}(2) \geq \dots \geq \mathcal{P}(n)$ we have $x \geq y$ and $\mathcal{P}(h) \geq \mathcal{P}(j)$ for all $h \leq j$. Inserting this into $\sum_{h=1}^n \mathcal{P}(h) = 1$ yields the stated inequalities. \square

In Figure 2 we illustrate these sets for $6 \leq n \leq 9$ and $\mathcal{P} \in \{\mathcal{BZ}, \mathcal{SS}\}$ in the following way. We have $\mathcal{A}_{1,2}^{\mathcal{P}} \subseteq [0, 1] \times [0, \frac{1}{2}]$ and color the pixels of $[0, 1] \times [0, \frac{1}{2}]$ in different colors. For (x, y) is ruled out by Lemma 1 we use red pixels and otherwise a scale of colors from dark blue to green. Here we use dark blue if (x, y) (actually all points in the near of (x, y)) is not contained in $\mathcal{A}_{1,2}^{\mathcal{P}}$. By light blue we denote that some elements of $\mathcal{A}_{1,2}^{\mathcal{P}}$ are in the near of (x, y) and by green we denote the case when there are *many* elements of $\mathcal{A}_{1,2}^{\mathcal{P}}$ are in the near of (x, y) . The value for voter 1 increases from left to right and the value for voter 2 increases from top to bottom. The graphics on the left hand side of Figure 2 correspond to the Banzhaf power index and those on the right hand side to the Shapley Shubik power index.

By a look at Figure 2 it seems that there might exist a certain subset of $[0, 1]^2$ which remains empty (or dark blue) if n tends to infinity and we factor out the intrinsic discrete nature of the problem. But of course proving such a property might be a very challenging task.

8. CONCLUSION

With the inverse power index problem we have presented a very challenging optimization problem of practical importance. Besides exhaustive enumeration we have developed the first exact algorithm to tackle this problem via an ILP formulation. The first computational results are very promising but of course a lot of work using more sophisticated techniques from integer linear (or non-linear) programming has to be done. We give a very precise

conjecture on the power distributions with the worst case approximation property and reveal some structure of the set of achievable power distributions by the Banzhaf and the Shapley Shubik power index. In our opinion it would be very challenging to prove these conjectures. Another open problem, which might be not too hard to solve, is to prove an analogue of Alon's and Edelman's result for the Shapley Shubik or other power indices. We leave open the problem to design a practical algorithm for the inverse power index problem, i. e. an algorithm combining fast primal heuristics and techniques for lower bounds in order to solve the problem either exactly or with a certain error guarantee.

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APPENDIX A. REGIONS WHICH ARE HARD TO APPROXIMATE

In Conjecture 1 and Conjecture 2 we state that the desired power distribution $d = (0.75, 0.25, 0, \dots)$ is the worst possible example with respect to approximation in the $\|\cdot\|_1$ -norm using one of the three presented voting methods and the Shapley-Shubik power index. In this section we want to show that there are further desired power distributions which are equally hard to approximate, at least for small n .

We restrict ourselves on complete simple games and weighted voting games, which coincide for $n \leq 5$ voters. For $n \leq 2$ and the Shapley-Shubik or the Banzhaf power index we only have the following achievable power distributions $s_1^1 = (1)$, $s_1^2 = (1, 0)$, and $s_2^2 = (0.5, 0.5)$ up to symmetry, i. e. we assume that the power is decreasing monotonely. Let $d = (d_1, \dots, d_n)$ be a theoretically possible desired power distribution, i. e. we have $d_i \geq 0$ and $\sum_{i=1}^n d_i = 1$. To avoid symmetric situations we assume $d_i \geq d_{i+1}$ for all $1 \leq i < n$ in the following.

Since there exists only one such theoretically possible desired power distribution we cannot say something interesting for $n = 1$ voters.

Lemma 2. For $n = 2$ and the Shapley-Shubik or the Banzhaf power index the regions where $\min_{\chi} \|d - \mathcal{P}(\chi)\|_1 = \frac{1}{2}$ are given by $d = (0.75, 0.25)$.

Proof. If $d_1 > 0.75$ we have $|1 - d_1| + |0 - d_2| < 0.25 + 0.25 = \frac{1}{2}$ and for $\frac{1}{2} \leq d_1 < 0.75$ we have $d_2 > 0.25$ and $|0.5 - d_1| + |0.5 - d_2| < 0.25 + 0.25 = \frac{1}{2}$. \square

For $n = 3$ voters the achievable power distributions for the Shapley-Shubik index up to symmetry are $s_1^3 = (1, 0, 0)$, $s_2^3 = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, $s_3^3 = (\frac{1}{2}, \frac{1}{2}, 0)$, and $s_4^3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Lemma 3. For $n = 3$ the regions where $\min_{\chi} \|d - \mathcal{SS}(\chi)\|_1 = \frac{1}{3}$ are given by

- $d_1 = \frac{5}{6}, \frac{1}{12} \leq d_2 \leq \frac{1}{6}$,
- $\frac{2}{3} \leq d_1 \leq \frac{5}{6}, d_1 + d_2 = 1$ (i.e. $d_3 = 0$),
- $\frac{1}{2} \leq d_1 \leq \frac{2}{3}, d_2 = \frac{1}{3}$,
- $d_1 = \frac{1}{2}, \frac{1}{4} \leq d_2 \leq \frac{1}{3}$, and
- $\frac{5}{12} \leq d_1 \leq \frac{1}{2}, d_1 + d_2 = \frac{5}{6}$ (i.e. $d_3 = \frac{1}{6}$).

Proof. (1) For $d_1 \geq \frac{5}{6}$ we have $\|s_1^3 - d\|_1 = 2 - 2d_1 \leq \frac{1}{3}$, where equality is attained iff $d_1 = \frac{5}{6}$.

(2.1) For $\frac{2}{3} \leq d_1 \leq \frac{5}{6}$ and $d_2 \leq \frac{1}{6}$ we have $\|s_2^3 - d\|_1 = 2d_1 - \frac{4}{3} \leq \frac{1}{3}$, where equality is attained iff $d_1 = \frac{5}{6}$.

(2.2) For $\frac{2}{3} \leq d_1 \leq \frac{5}{6}$ and $d_2 \geq \frac{1}{6}$ we have $d_3 \leq \frac{1}{6}$ and $\|s_2^3 - d\|_1 = \frac{1}{3} - 2d_3 \leq \frac{1}{3}$, where equality is attained iff $d_3 = 0$, i. e. $d_1 + d_2 = 1$.

(3.1) For $\frac{1}{2} \leq d_1 \leq \frac{2}{3}$ and $d_3 \geq \frac{1}{6}$ we have $\|s_2^3 - d\|_1 = \frac{4}{3} - 2d_1 \leq \frac{1}{3}$, where equality is attained iff $d_1 = \frac{1}{2}$.

(3.2) For $\frac{1}{2} \leq d_1 \leq \frac{2}{3}, d_3 \leq \frac{1}{6}$, and $d_2 \leq \frac{1}{3}$ we have $d_2 \geq \frac{1}{6}$ and $\|s_2^3 - d\|_1 = 2d_2 - \frac{1}{3} \leq \frac{1}{3}$, where equality is attained iff $d_2 = \frac{1}{3}$.

(3.3) For $\frac{1}{2} \leq d_1 \leq \frac{2}{3}$ and $d_2 \geq \frac{1}{3}$ we have $d_2 \leq \frac{1}{2}$ and $\|s_3^3 - d\|_1 = 1 - 2d_2 \leq \frac{1}{3}$, where equality is attained iff $d_2 = \frac{1}{3}$.

(4.1) For $\frac{5}{12} \leq d_1 \leq \frac{1}{2}$ and $d_3 \leq \frac{1}{6}$ we have $d_2 \leq \frac{1}{2}$ and $\|s_3^3 - d\|_1 = 2d_3 \leq \frac{1}{3}$, where equality is attained iff $d_3 = \frac{1}{6}$.

(4.2) For $\frac{5}{12} \leq d_1 \leq \frac{1}{2}$ and $d_2 \leq \frac{1}{3}$ we have $\|s_4^3 - d\|_1 = 2d_1 - \frac{2}{3} \leq \frac{1}{3}$, where equality is attained iff $d_1 = \frac{1}{2}$.

(4.3) For $\frac{5}{12} \leq d_1 \leq \frac{1}{2}, d_3 \geq \frac{1}{6}$, and $d_2 \geq \frac{1}{3}$ we have $d_3 \leq \frac{1}{4} < \frac{1}{3}$ and $\|s_4^3 - d\|_1 = \frac{2}{3} - 2d_3 \leq \frac{1}{3}$, where equality is attained iff $d_3 = \frac{1}{6}$.

(5.1) For $\frac{1}{3} \leq d_1 \leq \frac{5}{12}$ and $d_2 \geq \frac{1}{3}$ we have $\frac{1}{6} \leq d_3 \leq \frac{1}{3}$ and $\|s_4^3 - d\|_1 = \frac{2}{3} - 2d_3 \leq \frac{1}{3}$, where equality is attained iff $d_3 = \frac{1}{6}$.

(5.2) For $\frac{1}{3} \leq d_1 \leq \frac{5}{12}$ and $d_2 \leq \frac{1}{3}$ we have $\|s_4^3 - d\|_1 = 2d_1 - \frac{2}{3} < \frac{1}{3}$.

It remains to check, whether $\|s_i^3 - d\|_1 = \frac{1}{3}$ for the stated subspaces. This is easily verified. \square

For the Banzhaf power index and 3 voters the achievable power distributions up to symmetry are $b_1^3 = (1, 0, 0)$, $b_2^3 = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$, $b_3^3 = (\frac{1}{2}, \frac{1}{2}, 0)$, and $b_4^3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Lemma 4. For $n = 3$ the regions where $\min_{\chi} \|d - \mathcal{BZ}(\chi)\|_1 = \frac{2}{5}$ are given by $d_1 = \frac{4}{5}, \frac{1}{10} \leq d_2 \leq \frac{1}{5}$ and $d_1 = \frac{3}{4}, d_2 = \frac{1}{4}$.

Proof. (1) For $d_1 \geq \frac{4}{5}$ we have $\|b_1^3 - d\|_1 = 2 - 2d_1 \leq \frac{2}{5}$, where equality is attained iff $d_1 = \frac{4}{5}$.

(2.1) For $\frac{3}{5} \leq d_1 \leq \frac{4}{5}$ and $d_2 \geq \frac{1}{5}$ we have $d_3 \leq \frac{1}{5}$ and $\|b_2^3 - d\|_1 = \frac{2}{5} - 2d_3 \leq \frac{2}{5}$, where equality is attained iff $d_3 = 0$. So here we can proceed as in the proof of Lemma 2 to deduce $\min_{\chi} \|d - \mathcal{BZ}(\chi)\|_1 < \frac{2}{5}$ for

$$d_1 \neq \frac{3}{4}.$$

(2.2) For $\frac{3}{5} \leq d_1 \leq \frac{4}{5}$ and $d_2 \leq \frac{1}{5}$ we have $\|b_2^3 - d\|_1 = 2d_1 - \frac{6}{5} \leq \frac{2}{5}$, where equality is attained iff $d_1 = \frac{4}{5}$.

(3.1) For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$ and $d_2 \geq \frac{3}{10}$ we have $d_2 \leq \frac{1}{2}$ and $\|b_3^3 - d\|_1 = 1 - 2d_2 \leq \frac{2}{5}$, where equality is attained iff $d_2 = \frac{3}{10}$. For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$ and $d_2 = \frac{3}{10}$ we have $d_3 \leq \frac{1}{5}$ and $\|b_2^3 - d\|_1 = \frac{1}{5} < \frac{2}{5}$.

(3.2) For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}, d_2 \leq \frac{3}{10}$, and $d_3 \leq \frac{1}{5}$ we have $d_2 \geq \frac{1}{5}$ and $\|b_2^3 - d\|_1 = 2d_2 - \frac{2}{5} < \frac{2}{5}$.

(3.3) For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$ and $d_3 \geq \frac{1}{5}$ we have $\|b_3^3 - d\|_1 = \frac{6}{5} - 2d_1 < \frac{2}{5}$.

(4.1) For $d_1 \leq \frac{1}{2}$ and $d_3 \leq \frac{1}{5}$ we have $d_2 \leq \frac{1}{2}$ and $\|b_3^3 - d\|_1 = 2d_3 \leq \frac{2}{5}$, where equality is attained iff $d_3 = \frac{1}{5}$.

(4.2) For $\frac{2}{5} \leq d_1 \leq \frac{1}{2}$ and $d_3 \geq \frac{1}{5}$ we have $d_2 \geq \frac{1}{5}$ and $\|b_2^3 - d\|_1 = \frac{6}{5} - 2d_1 \leq \frac{2}{5}$, where equality is attained iff $d_1 = \frac{2}{5}$.

(5.1) For $d_1 \leq \frac{2}{5}$ and $d_2 \geq \frac{1}{3}$ we have $d_1 \geq \frac{1}{3}$, $\frac{1}{3} \geq d_3 \geq \frac{1}{5}$ and $\|b_4^3 - d\|_1 = \frac{2}{3} - 2d_3 < \frac{2}{5}$.

(5.2) For $d_1 \leq \frac{2}{5}$ and $d_2 \leq \frac{1}{3}$ we have $d_1 \geq \frac{1}{3}$ and $\|b_4^3 - d\|_1 = 2d_1 - \frac{2}{3} < \frac{2}{5}$.

For $d_3 = \frac{1}{5}$, see (4.1), we can assume $d_1 = d_2 = \frac{2}{5}$ due to (4.2) and can apply (5.1). It remains to check, whether $\|b_i^3 - d\|_1 = \frac{2}{5}$ for the stated subspaces. This is easily verified. \square

For $n = 4$ voters the achievable power distributions for the Shapley-Shubik index up to symmetry are $s_1^4 = (1, 0, 0, 0)$, $s_2^4 = (\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$, $s_3^4 = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0)$, $s_4^4 = (\frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12})$, $s_5^4 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$, $s_6^4 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, $s_7^4 = (\frac{5}{12}, \frac{5}{12}, \frac{1}{12}, \frac{1}{12})$, $s_8^4 = (\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12})$, $s_9^4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, $s_{10}^4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$, and $s_{11}^4 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

For the Banzhaf power index and 4 voters the achievable power distributions up to symmetry are $b_1^4 = (1, 0, 0, 0)$, $b_2^4 = (\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$, $b_3^4 = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, 0)$, $b_4^4 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$, $b_5^4 = (\frac{1}{2}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10})$, $b_6^4 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, $b_7^4 = (\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12})$, $b_8^4 = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$, $b_9^4 = (\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$, $b_{10}^4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, $b_{11}^4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$, and $b_{12}^4 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

Lemma 5. For $n = 4$ the regions where $\min_{\chi} \|d - \mathcal{BZ}(\chi)\|_1 = \frac{2}{5}$ are given by $d_1 = \frac{4}{5}$, $d_2 = \frac{1}{5}$ and $d_1 = \frac{3}{4}$, $d_2 = \frac{1}{4}$.

Proof. (1) For $d_1 \geq \frac{4}{5}$ we have $\|b_1^4 - d\|_1 = 2 - 2d_1 \leq \frac{2}{5}$, where equality is attained iff $d_1 = \frac{4}{5}$. If $d_1 = \frac{4}{5}$ and $d_2 \leq \frac{1}{10}$ then we have $d_4 \leq d_3 \leq \frac{1}{10}$ and $\|b_2^4 - d\|_1 = \frac{1}{5} < \frac{2}{5}$. If we have $d_1 = \frac{4}{5}$ and $d_2 \geq \frac{1}{10}$ then we have $d_2 \leq \frac{1}{5}$, $d_4 \leq d_3 \leq \frac{1}{10}$, and $\|b_2^4 - d\|_1 = 2d_2 \leq \frac{2}{5}$, where equality is attained iff $d_2 = \frac{1}{5}$ and $d_3 = d_4 = 0$.

(2.1) For $\frac{3}{5} \leq d_1 \leq \frac{4}{5}$ and $d_2 \geq \frac{1}{5}$ we have $\|b_3^4 - d\|_1 = \frac{2}{5} - 2d_3 \leq \frac{2}{5}$, where equality is attained iff $d_3 = d_4 = 0$. So here we can proceed as in the proof of Lemma 2 to deduce $\min_{\chi} \|d - \mathcal{BZ}(\chi)\|_1 < \frac{2}{5}$ for $d_1 \neq \frac{3}{4}$.

(2.2) For $\frac{3}{5} \leq d_1 \leq \frac{4}{5}$, $d_2 \leq \frac{1}{5}$, and $d_1 + d_4 \leq \frac{4}{5}$ we have $\|b_3^4 - d\|_1 = 2d_1 + 2d_4 - \frac{6}{5} \leq \frac{2}{5}$, where equality is attained iff $d_1 + d_4 = \frac{4}{5}$.

(2.3) For $\frac{3}{5} \leq d_1 \leq \frac{4}{5}$, $d_2 \leq \frac{1}{5}$, and $d_1 + d_4 \geq \frac{4}{5}$ we have $d_4 \leq \frac{1}{10}$, $d_1 \geq \frac{7}{10}$, and $\|b_2^4 - d\|_1 \leq \frac{2}{5}$, where equality is only possible for $d_4 = 0$ so that (2.2) yields $d_1 = \frac{4}{5}$ and we can apply (1).

(3.1) For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$ and $d_2 \geq \frac{3}{10}$ we have $d_2 \leq \frac{1}{2}$ and $\|b_4^4 - d\|_1 = 1 - 2d_2 \leq \frac{2}{5}$, where equality is attained iff $d_2 = \frac{3}{10}$. For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$ and $d_2 = \frac{3}{10}$ we have $d_3 + d_4 \leq \frac{1}{5}$, $d_4 \leq \frac{1}{10}$, and $\|b_3^4 - d\|_1 = \frac{1}{5} + 2d_4 \leq \frac{2}{5}$ where equality is attained iff $d_3 = d_4 = \frac{1}{10}$ and $d_1 = \frac{1}{2}$. But in the latter case we have $\|b_5^4 - d\|_1 = 0$.

(3.2) For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$, $\frac{1}{5} \leq d_2 \leq \frac{3}{10}$, and $d_3 \leq \frac{1}{5}$ we have $d_3 + d_4 = 1 - d_1 - d_2 \geq \frac{7}{10} - d_1$ so that $d_3 \geq \frac{7}{20} - \frac{d_1}{2}$ holds. Thus we have $\|b_3^4 - d\|_1 = \frac{8}{5} - 2d_1 - 2d_3 \leq \frac{9}{10} - d_1 \leq \frac{2}{5}$, where equality is attained iff $d_1 = \frac{1}{2}$, $d_2 = \frac{3}{10}$, $d_3 = d_4 = \frac{1}{10}$, which equals b_5^4 .

(3.3) For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$ and $d_3 \geq \frac{1}{5}$ we have $d_2 \geq \frac{1}{5}$ and $\|b_3^4 - d\|_1 = \frac{6}{5} - 2d_1 < \frac{2}{5}$.

(3.4) For $\frac{1}{2} \leq d_1 \leq \frac{3}{5}$ and $d_2 \leq \frac{1}{5}$ we have $d_3 \leq \frac{1}{5}$ and $\|b_3^4 - d\|_1 = 2d_4 < \frac{2}{5}$ since $d_4 \leq \frac{1-d_1}{3} \leq \frac{1}{6}$.

(4.1) For $d_1 \leq \frac{1}{2}$ and $d_3 + d_4 \leq \frac{1}{5}$ we have $d_2 \leq \frac{1}{2}$ and $\|b_4^4 - d\|_1 = 2d_3 + 2d_4 \leq \frac{2}{5}$, where equality is attained iff $d_3 + d_4 = \frac{1}{5}$. For the latter case see (4.7).

(4.2) For $\frac{2}{5} \leq d_1 \leq \frac{1}{2}$ and $d_3 \geq \frac{1}{5}$ we have $d_2 \geq \frac{1}{5}$ and $\|b_3^4 - d\|_1 = \frac{6}{5} - 2d_1 \leq \frac{2}{5}$, where equality is attained iff $d_1 = \frac{2}{5}$. For the latter case see (5.1)-(5.4).

(4.3) For $d_1 \leq \frac{1}{2}$, $d_2 \leq \frac{11}{30}$, and $d_3 \leq \frac{1}{6}$ we have $d_2 \geq \frac{1}{6}$, $d_4 \leq \frac{1}{6}$, and $\|b_6^4 - d\|_1 = 2d_2 - \frac{1}{3} \leq \frac{2}{5}$, where equality is attained iff $d_2 = \frac{11}{30}$. In the latter case we consider $x = \|b_5^4 - d\|_1$. For $d_4 \leq d_3 \leq \frac{1}{10}$ we have $x = \frac{2}{15} < \frac{2}{5}$ and for $d_3 \geq d_4 \geq \frac{1}{10}$ we have $x = 2d_3 + 2d_4 - \frac{4}{15} \leq \frac{2}{5}$, where equality is attained iff $d_3 + d_4 = \frac{1}{3}$, i. e. $d_3 = d_4 = \frac{1}{6}$, and $\|b_{11}^4 - d\|_1 < \frac{2}{5}$. Finally, for $d_3 \geq \frac{1}{10}$ and $d_4 \leq \frac{1}{10}$ we have $x = 2d_3 - \frac{1}{15} < \frac{2}{5}$.

(4.4) For $\frac{2}{5} \leq d_1 \leq \frac{1}{2}$ and $d_4 \geq \frac{1}{6}$ we have $d_2 \geq d_3 \geq \frac{1}{6}$ and $\|b_6^4 - d\|_1 = 1 - 2d_1 < \frac{2}{5}$.

(4.5) For $\frac{2}{5} \leq d_1 \leq \frac{1}{2}$, $d_2 \geq \frac{1}{5}$, $d_3 \leq \frac{1}{5}$, and $d_3 + d_4 \geq \frac{1}{5}$ we have $d_4 \leq \frac{1}{5}$ and $\|b_8^4 - d\|_1 = \frac{4}{5} - 2d_3 - 2d_4 \leq \frac{2}{5}$, where equality is attained iff $d_3 + d_4 = \frac{1}{5}$. For the latter case see (4.7).

(4.6) For $\frac{2}{5} \leq d_1 \leq \frac{1}{2}$ and $d_2 \leq \frac{1}{5}$ we have $d_4 \leq d_3 \leq \frac{1}{5}$ and $\|b_8^4 - d\|_1 = 2d_1 - \frac{4}{5} < \frac{2}{5}$.

(4.7) For $\frac{2}{5} \leq d_1 \leq \frac{1}{2}$ and $d_3 + d_4 = \frac{1}{5}$ we have $d_2 \geq \frac{3}{10}$, $d_3 \geq \frac{1}{10}$, $d_4 \leq \frac{1}{10}$, $d_2 + d_3 \leq 1 - d_1 \leq \frac{3}{5}$ and $\|b_5^4 - d\|_1 = 2d_2 + 2d_3 - \frac{4}{5} \leq \frac{2}{5}$, where equality is attained iff $d_2 + d_3 = \frac{3}{5}$, i. e. $d_1 = \frac{2}{5}$, $d_2 = \frac{2}{5}$, $d_3 = \frac{1}{5}$, and $d_4 = 0$. In the latter case we have $\|b_{10}^4 - d\|_1 < \frac{2}{5}$.

- (5.1) For $d_1 \leq \frac{2}{5}$, $d_2 \geq \frac{1}{3}$, and $d_3 \leq \frac{3}{10}$ we have $d_1 \geq \frac{1}{3}$, $d_2 \leq \frac{2}{5}$, $d_4 \leq \frac{1}{6}$ and $\|b_9^4 - d\|_1 = |d_1 - \frac{3}{8}| + |d_2 - \frac{3}{8}| + |d_3 - \frac{1}{8}| + |d_4 - \frac{1}{8}| \leq \frac{1}{24} + \frac{1}{24} + \frac{7}{40} + \frac{1}{8} < \frac{2}{5}$.
- (5.2) For $d_1 \leq \frac{2}{5}$, $d_2 \geq \frac{1}{3}$, and $d_3 \geq \frac{3}{10}$ we have $d_1 \geq \frac{1}{3}$, $d_3 \leq \frac{1}{3}$, and $\|b_{10}^4 - d\|_1 = \frac{2}{3} - 2d_3 < \frac{2}{5}$.
- (5.3) For $d_1 \leq \frac{2}{5}$, $d_2 \leq \frac{1}{3}$, and $d_4 < \frac{2}{15}$ we have $d_1 > \frac{13}{45}$. For $d_1 \geq \frac{1}{3}$ we have $\|b_{10}^4 - d\|_1 = 2d_1 + 2d_4 - \frac{2}{3} < \frac{2}{5}$ and for $d_1 < \frac{1}{3}$ we have $\|b_{10}^4 - d\|_1 = 2d_4 < \frac{2}{5}$.
- (5.4) For $\frac{7}{24} \leq d_1 \leq \frac{2}{5}$, $d_2 \leq \frac{1}{3}$, and $d_4 \geq \frac{2}{15}$ we have $d_2 \geq \frac{1}{5}$ and consider $x = \|b_{11}^4 - d\|_1$. If $d_3 \geq d_4 \geq \frac{1}{6}$ then we have $x = \frac{2}{3} - 2d_2 < \frac{2}{5}$ for $d_1 \geq \frac{1}{3}$ and $x = \frac{4}{3} - 2d_1 - 2d_2 < \frac{2}{5}$. If $d_4 \leq d_3 \leq \frac{1}{6}$ then we have $x = 2d_1 - \frac{2}{3} < \frac{2}{5}$ for $d_1 \geq \frac{1}{3}$ and $x = 0$ for $d_1 \leq \frac{1}{3}$. If $d_3 \geq \frac{1}{6}$ and $d_4 \leq \frac{1}{6}$ then we have $x = 2d_1 + 2d_3 - 1 < \frac{2}{5}$ for $d_1 \geq \frac{1}{3}$ and $x = 2d_3 - \frac{1}{3}$ for $d_1 \leq \frac{1}{3}$.
- (6) For $\frac{1}{4} \leq d_1 \leq \frac{7}{24}$ we have $d_3 \leq d_2 \leq \frac{7}{24}$, $d_4 = 1 - d_1 - d_2 - d_3 \geq \frac{1}{8}$ and we consider $x = \|b_{12}^4 - d\|_1$. For $d_4 \leq d_3 \leq d_2 \leq \frac{1}{4}$ we have $x = 2d_1 - \frac{1}{2} < \frac{2}{5}$, for $d_2 \geq \frac{1}{4}$ and $d_4 \leq d_3 \leq \frac{1}{4}$ we have $x = 2d_1 + 2d_2 - 1 < \frac{2}{5}$, and for $d_2 \geq d_3 \geq \frac{1}{4}$ and $d_4 \leq \frac{1}{4}$ we have $x = \frac{1}{2} - 2d_4 < \frac{2}{5}$.

Finally we have to show that $\min_{\chi} \|d - \mathcal{BZ}(\chi)\|_1 = \frac{2}{5}$ for the stated regions. For $d_1 = 0.75$, $d_2 = 0.25$ we have done this already by solving an ILP. Now let $d_1 = \frac{4}{5}$ and $d_2 = \frac{1}{5}$. We can easily check $\|d - \mathcal{BZ}(b_i^4)\|_1 > \frac{2}{5}$ for all $1 \leq i \leq 12$ and $\|d - \mathcal{BZ}(b_1^4)\|_1 = \frac{2}{5}$. \square

We would like to remark that the weighted voting game $\chi = [5; 4, 1, 1, 1, 1]$ has $(\frac{15}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19})$ as its Banzhaf vector and fulfills $\|(0.8, 0.2, 0, 0, 0) - \mathcal{BZ}(\chi)\|_1 < \frac{1}{3}$. So the desired power distribution $(0.8, 0.2, 0, 0, 0)$ is not an example for worst possible approximation property. So maybe for $n \geq 5$ the desired power distribution $(0.75, 0.25, 0, 0, 0)$ is the unique example with worst possible approximation property for the Banzhaf power index. We remark $\|(0.75, 0.25, 0, 0, 0) - \mathcal{BZ}(\chi)\|_1 = \frac{30}{76} \approx 0.3947368$.

As far as the Shapley Shubik power index is concerned we have no idea if $(0.75, 0.25, 0, \dots)$ is the unique example with worst possible approximation property for large n or if there are regions which are hard to approximate. In principle we can utilize our ILP formulation from Section 4 to check whether a whole region like $d_1 = \frac{5}{6}$, $\frac{1}{12} \leq d_2 \leq \frac{1}{6}$ can not be approximated with an error less than $\frac{1}{3}$ by letting the d_i variables with certain range bounds instead of numerical constants. As an example we have tried this for $n = 6$ and obtained the weighted voting game $\chi = [5; 5, 1, 1, 1, 1, 1]$ approximating the desired power distribution $d = (\frac{5}{6}, \frac{1}{12}, \frac{1}{12}, 0, 0, 0)$ with an error of $\|d - \mathcal{SS}(\chi)\|_1 = \frac{1}{5} < \frac{1}{3}$.

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